

# On the Tanner Graph Cycle Distribution of Random LDPC, Random Protograph-Based LDPC, and Random Quasi-Cyclic LDPC Code Ensembles

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## Abstract

Random bipartite graphs, random lifts of bipartite protographs, and random cyclic lifts of bipartite protographs are used to represent random low-density parity-check (LDPC) codes, randomly constructed protograph-based LDPC codes, and random quasi-cyclic (QC) LDPC codes, respectively. In this paper, we study the distribution of cycles of different length in all these three categories of graphs. We prove that for a random bipartite graph, with a given degree distribution, the distributions of cycles of different length tend to independent Poisson distributions, as the size of the graph tends to infinity. We also derive the expected values of the Poisson distributions, and show that they are independent of the size of the graph, and only depend on the degree distribution and the cycle length. It is well-known that for a random lift of a protograph, the distributions of cycles of different length  $c$  tend to independent Poisson distributions with expected value equal to the number of tailless backtrackless closed (tbc) walks of length  $c$  in the protograph, as the size of the graph (lifting degree) tends to infinity. Here, we find the number of tbc walks in a bi-regular protograph, and demonstrate that random bi-regular LDPC codes have essentially the same cycle distribution as random protograph-based LDPC codes, as long as the degree distributions are identical. For random QC-LDPC codes, however, we show that the cycle distribution can be quite different from the other two categories. While for the former categories, the expected number of cycles of different length is  $\Theta(1)$  with respect to the size of the graph, for the case of QC-LDPC codes, depending on the protograph and the value of  $c$ , it can be either  $\Theta(N)$  or  $\Theta(1)$ , where  $N$  is the lifting degree (code length). For QC-LDPC codes, we also derive an upper bound on the variance of the number of cycles of different length. This bound increases linearly with  $N$ . In addition, we provide numerical results that match our theoretical derivations. Our results also provide a theoretical foundation for empirical results that were reported in the literature but were not well-justified.

**Index Terms:** Low-density parity-check (LDPC) codes, random LDPC codes, quasi cyclic (QC) LDPC codes, protograph-based LDPC codes, cycle distribution of LDPC codes, lifting, cyclic lifting.

## I. INTRODUCTION

The performance of low-density parity-check (LDPC) codes under iterative message-passing algorithms is highly dependent on the structure of the code's Tanner graph, in general, and the distribution of short cycles, in particular, see, e.g., [1], [2], [3], [4]. The cycles play a particularly important role in the error floor performance of LDPC codes, where they form the main substructure of the trapping sets [5], [6], [7], [8], [9].

Counting and enumerating (finding) cycles of a given length in a general graph is known to be NP-hard [10]. (For a rather comprehensive literature review on algorithms to count and enumerate cycles in different types of graphs, including bipartite graphs, and their complexity, the reader is referred to [11].) It is thus of interest to have simple approximations for the number of cycles of a given length in a given graph. Related to this, it is also interesting to obtain the distribution of cycles of a given length in an ensemble of Tanner graphs (LDPC codes). The knowledge of such a distribution, including the expected value and variance, can help in the analysis and in guiding the design of LDPC codes. The expected value can also be used as an approximation for the number of cycles of a given length in a given graph in the ensemble, with the variance providing a measure of accuracy of the approximation.

In [12], Bollobás showed that, for a given random graph with an arbitrary degree distribution and a fixed  $c$ , as the size of the graph tends to infinity, the multiplicities of cycles of lengths  $3, 4, 5, \dots, c$ , tend to independent Poisson random variables. He also derived the expected values of the random variables. Later, in [13], the authors considered random bipartite graphs, in which all the nodes have the same degree  $d$ , and  $c$  can grow as a function of the number nodes in the graph, and proved that as the size of the graph tends to infinity, the distributions of cycles of different length  $c$  tend to independent Poisson distributions with expected values  $\mu = (d-1)^c/c$ .

In this work, we consider the case of random bipartite graphs with arbitrary degree distributions  $\{d_i\}$  and  $\{d'_i\}$  on the two parts of the graph, respectively, and prove that the multiplicities of cycles of different length  $c$ , as the size of the graph tends to infinity, tend to independent Poisson

random variables with the following expected values:

$$\mu = \frac{\left(\left(\frac{2}{|E|} \sum_{i=1}^n \binom{d_i}{2}\right)\left(\frac{2}{|E|} \sum_{i=1}^m \binom{d'_i}{2}\right)\right)^{c/2}}{c}, \quad (1)$$

where  $n$  and  $m$  are the number of nodes in the two parts of the graph, and  $|E|$  is the number of edges of the graph. Unlike the bipartite graphs studied in [13], the graphs studied in this work are those representing (irregular and bi-regular) LDPC codes. For the special case of bi-regular LDPC codes, Equation (1) reduces to

$$\mu = \frac{\left((d_u - 1)(d_w - 1)\right)^{c/2}}{c}, \quad (2)$$

in which  $d_u$  and  $d_w$  denote the degrees of nodes in the two parts of the graph. Equation (2) implies that, at sufficiently large block lengths, the average number of cycles, as well as the variances, do not depend on the block length of the code. This matches the observation made in [11] through numerical results.

The construction of LDPC codes by lifting a small bipartite graph, called *base graph* or *protograph*, was first appeared in [14]. Since then, there has been a flurry of research activity on the analysis and design of protograph-based LDPC codes, see, e.g., [15], [16], and the references therein. Protograph-based LDPC codes are interesting as they lend themselves to a more compact representation, which in turn, translates to simpler encoding and decoding algorithms. A particularly popular category of protograph-based LDPC codes are those constructed by *cyclic liftings* [17], [18], [19], [20], [21], [22], [6]. Such codes are quasi cyclic (QC), and are the most popular in practice, as their implementation is even simpler. For that reason, they have also been adopted in a number of standards [23], [24].

It was shown by Fortin and Rudinsky in [25] that for a random lift of a protograph, the distributions of cycles of different length tend to independent Poisson distributions as the size of the graph tends to infinity. They also showed that the expected value of the number of cycles of length  $c$  is equal to  $T(G, c)$ , where  $T(G, c)$  is the number of tailless backtrackless closed walks (tbc walks) of length  $c$  in the protograph. In this work, we calculate  $T(G, c)$  for bi-regular protographs, in general, and fully-connected bipartite protographs, in particular. Using these results, we show that the cycle distributions of random bi-regular graphs and those of random

lifts of a bi-regular protograph with a similar degree distribution are essentially identical, in the asymptotic regime where the graph size tends to infinity.

In [20], an efficient algorithm for counting short cycles in the Tanner graph of a QC-LDPC code is proposed. Using numerical results, it was shown in [20], that randomly constructed QC-LDPC codes have a much better girth distribution compared to their counterparts that lack the QC structure. In this work, by viewing the Tanner graphs of QC-LDPC codes as cyclic lifts of protographs, we study their cycle distribution. We demonstrate that the cycle distributions for random cyclic lifts of a bipartite protograph can be quite different from those of random bipartite graphs and random lifts of bipartite protographs of similar degree distributions. In particular, we show that depending on the protograph and the cycle length  $c$ , the expected value of the number of cycles of length  $c$  in random cyclic lifts can increase linearly with the size of the graph. This is while for random bipartite graphs and random lifts of bipartite protographs, the expected number of cycles of length  $c$  remains constant with increase in the graph size, regardless of the value of  $c$  or the choice of protograph or degree distribution. These results explain the differences observed in [20] regarding the cycle distributions of QC-LDPC codes versus LDPC codes that lack the QC structure.

The organization of the rest of the paper is as follows: In Section II, we present some definitions and notations. This is followed in Section III by our results on the cycle distribution of random LDPC codes. In Section IV, we discuss the cycle distribution of random lifts of a protograph, and calculate  $T(G, c)$  for bi-regular protographs. The results on the expected value and the variance of the number of cycles for QC-LDPC codes are presented in Section V. Section VI is devoted to numerical results. The paper is concluded with some remarks in Section VII.

## II. DEFINITIONS AND NOTATIONS

An undirected graph  $G = (V, E)$  is defined as a set of vertices or nodes  $V$  and a set of edges  $E$ , where  $E$  is a subset of the pairs  $\{\{u, v\} : u, v \in V, u \neq v\}$ . In this work, we consider graphs with no loop or parallel edges. A *walk* of length  $k$  in the graph  $G$  is a sequence of nodes  $v_1, v_2, \dots, v_{k+1}$  in  $V$  such that  $\{v_i, v_{i+1}\} \in E$ , for all  $i \in \{1, \dots, k\}$ . Equivalently, a walk of length  $k$  can be described by the corresponding sequence of  $k$  edges. A walk is a *path* if all the nodes  $v_1, v_2, \dots, v_k$  are distinct. A walk is called a *closed walk* if the two end nodes are identical, i.e., if  $v_1 = v_{k+1}$ . Under the same condition, a path is called a *cycle*. We denote cycles

of length  $k$ , also referred to as  $k$ -cycles, by  $C_k$ . We use  $N_k$  for  $|C_k|$ . Consider a walk  $\mathcal{W}$  of length  $k$  represented by the sequence of edges  $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ . The walk  $\mathcal{W}$  is *backtrackless*, if  $e_{i_s} \neq e_{i_{s+1}}$ , for any  $s \in \{1, \dots, k-1\}$ . Also, the walk  $\mathcal{W}$  is *tailless*, if  $e_{i_1} \neq e_{i_k}$ . In this paper, we use the term *tbc walk* to refer to a tailless backtrackless closed walk.

A graph  $G = (V, E)$  is called *bipartite*, if the node set  $V$  can be partitioned into two disjoint subsets  $U$  and  $W$ , i.e.,  $V = U \cup W$  and  $U \cap W = \emptyset$ , such that every edge in  $E$  connects a node from  $U$  to a node from  $W$ . Tanner graphs of LDPC codes are bipartite graphs, in which  $U$  and  $W$  are referred to as *variable nodes* and *check nodes*, respectively. Parameters  $n$  and  $m$  in this case are used to denote  $|U|$  and  $|W|$ , respectively. Parameter  $n$  is the code's block length and the code rate  $R$  satisfies  $R \geq 1 - (m/n)$ .

The number of edges connected to a node  $v$  is called the *degree* of the node  $v$ , and is denoted by  $d_v$  (or  $\deg(v)$ ). We call a bipartite graph  $G = (U \cup W, E)$  *bi-regular*, if all the nodes on the same side of the given bipartition have the same degree, i.e., if all the nodes in  $U$  have the same degree  $d_u$  and all the nodes in  $W$  have the same degree  $d_w$ . Note that, for a bi-regular graph,  $|U|d_u = |W|d_w = |E|$ , where  $|E|$  is the number of edges in the bipartite graph. A bipartite graph that is not bi-regular is called *irregular*. We call a bipartite graph *fully-connected* or *complete*, if it is bi-regular and if  $d_u = |W|$  and  $d_w = |U|$ . A (non-bipartite) graph is called *complete* if every node is connected to all the other nodes. We use the notation  $K_a$  for a complete graph with  $a$  nodes.

The *adjacency matrix* of a graph  $G$  is the matrix  $A = [a_{ij}]$ , where  $a_{ij}$  is the number of edges connecting the node  $i$  to the node  $j$  for all  $i, j \in V$ . Matrix  $A$  is symmetric and since we have assumed that  $G$  has no parallel edges or loops,  $a_{ij} \in \{0, 1\}$  for all  $i, j \in V$ , and  $a_{ii} = 0$  for all  $i \in V$ . One important property of the adjacency matrix that we will use for our results is that the number of walks between any two nodes of the graph can be determined using the powers of this matrix. More precisely, the entry in the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of  $A^k$ ,  $[A^k]_{ij}$ , is the number of walks of length  $k$  between nodes  $i$  and  $j$ . In particular,  $[A^k]_{ii}$  is the number of closed walks of length  $k$  containing the node  $i$ .

Let  $G(V = U \cup W, E)$  be a bipartite graph with  $|U| = n$  and  $|W| = m$ , and consider an assignment of a permutation  $\pi^e \in S_N$  to each edge  $e$  in  $E$ , where  $S_N$  is the symmetric group over  $\mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}$ . Consider the following construction of the graph  $\tilde{G}(\tilde{V}, \tilde{E})$  from  $G(V, E)$ : We make  $N$  copies of  $G$  such that for each node  $v \in V$ , we have a set of

nodes  $\tilde{v} = \{v^0, \dots, v^{N-1}\}$  in  $\tilde{V}$ . Similarly, for each edge  $e = \{u, w\} \in E$ , we have a set of edges  $\tilde{e} = \{e^0, \dots, e^{N-1}\}$  in  $\tilde{E}$  such that  $\{u^i, w^j\}$  belongs to  $\tilde{E}$  if and only if  $\pi^e(i) = j$ . In this construction, graph  $\tilde{G}$  is called an  $N$ -*lifting* of  $G$ . Graph  $G$  is called the *base graph* or *protograph*, and the parameter  $N$  is referred to as the *lifting degree*. The lifted graph  $\tilde{G}$  can be considered as the Tanner graph of an LDPC code  $\tilde{C}$ , i.e., the parity-check matrix  $\tilde{H}$  of  $\tilde{C}$  is defined to be the incidence matrix of  $\tilde{G}$ . The code  $\tilde{C}$ , in this case, is called the *lifted code*, and the incidence matrix  $H$  of  $G$  is called the *base matrix*. The  $mN \times nN$  parity-check matrix  $\tilde{H}$  of  $\tilde{C}$  consists of  $m \times n$  submatrices  $[\tilde{H}]_{ij}$ ,  $0 \leq i \leq m-1$ ,  $0 \leq j \leq n-1$ , where each submatrix is a permutation matrix of size  $N \times N$ , if the entry  $[H]_{ij} \neq 0$ ; otherwise,  $[\tilde{H}]_{ij}$  is the all-zero matrix. The LDPC codes constructed by the lifting process, just explained, are referred to as *protograph-based LDPC codes*. In the lifting process, if the permutations are selected randomly from  $S_N$ , the constructed codes are called *random lifts*.

Consider the subgroup  $C_N$  of symmetric group  $S_N$  over  $\mathbb{Z}_N$ , where  $C_N$  contains all circulant permutations  $\pi_p$ . The index  $p$  of the permutation  $\pi_p$  corresponds to  $p$  cyclic shifts to the left. If the permutations in the lifting process are cyclic, i.e., if they are selected from  $C_N$ , then the resulting graph  $\tilde{G}$  is called a *cyclic lift* of  $G$ , and the associated code is quasi-cyclic (QC). In this case, the non-zero submatrices of  $\tilde{H}$  are circulant permutation matrices (CPM). In particular, when the entry  $[H]_{ij} \neq 0$ , then  $[\tilde{H}]_{ij} = I^{p_{ij}}$ ,  $p_{ij} \in \mathbb{Z}_N$ , where  $I^{p_{ij}}$  is a CPM whose rows are obtained by cyclically shifting the rows of the identity matrix to the left by  $p_{ij}$ . We also take  $I^{+\infty}$  to represent the all-zero matrix. We refer to the  $m \times n$  matrix  $P = [p_{ij}]; 0 \leq i \leq m-1, 0 \leq j \leq n-1$ , as the *permutation shift matrix* or the *exponent matrix* corresponding to the lifted code  $\tilde{C}$  or to the lifted graph  $\tilde{G}$ . Clearly, there is a one-to-one correspondence between  $P$  and  $\tilde{H}$ .

Consider a QC-LDPC code  $\tilde{C}$  corresponding to an exponent matrix  $P$ . It is well-known that a necessary condition for the existence of a cycle of length  $2k$  in the Tanner graph of  $\tilde{C}$ , corresponding to  $\tilde{H}$ , is

$$\sum_{i=0}^{k-1} (p_{m_i, n_i} - p_{m_i, n_{i+1}}) = 0 \pmod{N}, \quad (3)$$

where  $n_k = n_0$ ,  $m_i \neq m_{i+1}$ ,  $n_i \neq n_{i+1}$ , and none of the permutation shifts in (3) is  $+\infty$  [17]. The sequence of permutation shifts in (3) corresponds to a tbc walk in the base graph, i.e., cycles of the lifted graph are the inverse images of tbc walks with zero permutation shift in the base graph [26], [22]. In fact, an additional requirement for the sequence of permutation shifts in

(3) to correspond to a cycle in the lifted graph is that no subsequence of the permutation shifts should correspond to a tbc walk of permutation shift zero in the base graph [26], [22]. For a tbc walk  $w$  in  $G$ , we refer to the summation in (3) as the permutation shift corresponding to  $w$  and denote it by  $\mathcal{P}(w)$ . It is clear that depending on the starting index  $n_0$  of  $w$ , or the direction of travel along  $w$ , the sign of  $\mathcal{P}(w)$  may change. As we are only concerned about the value of  $\mathcal{P}(w)$  being zero or non-zero, in the context of this work, the two values  $\pm\mathcal{P}(w)$  are considered equivalent.

It is well-known that there are cycles in cyclic lifts of a base graph that are independent of the lifting degree  $N$  or the choice of the exponent matrix  $P$  [17], [18]. Such cycles, referred to as *inevitable cycles*, occur if there exists a tbc walk  $w$  in the base graph, in which, each edge is traversed in both directions equal number of times. In this case,  $\mathcal{P}(w) = 0 \pmod{N}$ , regardless of the value of  $N$ , or the choice of  $P$ . Such tbc walks are referred to as *zero-permutation (ZP) tbc walks* in this paper. We also use the terminology *prime ZP tbc walk* for a ZP tbc walk that does not contain any ZP tbc subwalk. In fact, inevitable cycles in the lifted graph are the inverse images of prime ZP tbc walks in the base graph. Clearly, inevitable cycles (prime ZP tbc walks) only depend on the structure of the base graph.

### III. RANDOM IRREGULAR AND BI-REGULAR GRAPHS

In the following, we prove our result on the cycle distribution of random irregular bipartite graphs with arbitrary degree distributions. Although many of the general steps of the proof are similar to those taken in [12] to prove the result on the cycle distribution of a (non-bipartite) random graph, there are major differences in the details.

**Theorem 1.** *Let  $\Delta$  be a fixed natural number satisfying  $\Delta \geq d_1 \geq d_2 \geq \dots \geq d_n$ , and  $\Delta \geq d'_1 \geq d'_2 \geq \dots \geq d'_m$ , where  $\sum_{i=1}^n d_i = \sum_{i=1}^m d'_i = \eta$ . Also, let  $2\eta - n \rightarrow \infty$  and  $2\eta - m \rightarrow \infty$  as  $n, m \rightarrow \infty$ . Consider the probability space  $\mathcal{G}$  of all bipartite graphs with node set  $(U, W)$ , where  $U = \{u_1, u_2, \dots, u_n\}$ ,  $W = \{w_1, w_2, \dots, w_m\}$ , and in which the degree of node  $u_i$  is  $d_i$  and the degree of node  $w_i$  is  $d'_i$ . Suppose that the graphs in  $\mathcal{G}$  are selected uniformly at random. For  $G \in \mathcal{G}$ , denote by  $N_i(G)$  the number of cycles of length  $i$  in  $G$ . Then, for any fixed even value of  $k \geq 4$ , the random variables  $N_4, \dots, N_k$ , are asymptotically independent*

Poisson random variables with  $N_c$  having the expected value

$$E(N_c) \simeq \frac{\left(\frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2}\right) \left(\frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2}\right)^{c/2}}{c}.$$

*Proof:* In a random bipartite graph, for each node  $z$ , we consider a bin that contains  $\deg(z)$  cells. We now consider a random perfect matching to pair the cells on the  $U$  side of the graph to the cells on the  $W$  side. Corresponding to each matching, we construct a bipartite graph such that if there is an edge between two cells, then we place an edge between the corresponding nodes (bins) in the bipartite graph. The bipartite graphs are thus represented as images of the so-called *configurations* that are obtained from the random perfect matchings. Clearly, there are  $N(\eta) = \eta!$  configurations, where  $\eta$  is the number of edges in the graph. We consider a probability space where the configurations are selected uniformly at random. Since we are interested in bipartite graphs with no parallel edges, we are only concerned with configurations that have at most one connection between any two bins (nodes). It can be shown that the probability of configurations that violate this property goes to zero asymptotically. The proof is similar to that of [12] for random non-bipartite graphs and is thus omitted.

For a configuration, we define a cycle of length  $k$  to be a set of  $k$  edges, like  $\{e_1, e_2, \dots, e_k\}$ , such that there are  $k$  distinct bins, like  $D_{i_1}, \dots, D_{i_k}$ , such that for each  $j \in \{1, \dots, k\}$ , the edge  $e_j$ , connects a cell in bin  $D_{i_j}$  to a cell in bin  $D_{i_{j+1}}$ , where  $D_{i_{k+1}} = D_{i_1}$ , and the two cells in each bin  $D_{i_j}$ , connected to the two edges  $e_j$  and  $e_{j-1}$ , are distinct ( $e_0 = e_k$ ). We now compute the number of  $k$ -cycles,  $\mathcal{C}_k$ , in a configuration. To form a  $k$ -cycle, one needs to choose  $k/2$  bins from  $U$  and  $k/2$  bins from  $W$ . Next, from each bin, one needs to choose two cells (the order of the two cells is important). Suppose that bin  $i$  contains  $d_i$  cells. We thus have  $(d_i)(d_i - 1)$  choices for the two cells of bin  $i$ . Hence, in order to choose all the cells on both sides of the graph, we have

$$\left( \sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left( \sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right)$$

choices. To count the number of  $k$ -cycles in a configuration, we also need to consider different orderings of the  $k/2$  bins on each side of the graph. This results in

$$\mathcal{C}_k = \left( \sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left( \sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left( \frac{(\frac{k}{2})! (\frac{k}{2})!}{k} \right), \quad (4)$$



where the division by  $k$  is for counting each cycle in the above process  $k$  times.

We note that given a set of  $\ell$  fixed edges, there are  $(\eta - \ell)!$  configurations containing those edges. The image of a  $k$ -cycle in a configuration is a  $k$ -cycle in the graph. We thus have

$$\begin{aligned}
E(N_c) &\simeq \frac{\mathcal{C}_c \times (\eta - c)!}{\eta!} \\
&= \left( \sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left( \sum_{\substack{\sigma \subset W \\ |\sigma|=c/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left( \frac{(\frac{c}{2})! (\frac{c}{2})!}{c} \right) \times \frac{(\eta - c)!}{\eta!} \\
&\simeq \left( \sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left( \sum_{\substack{\sigma \subset W \\ |\sigma|=c/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left( \frac{(\frac{c}{2})! (\frac{c}{2})!}{c} \right) \times \frac{1}{\eta^c}. \tag{5}
\end{aligned}$$

In order to simplify Equation (5), we will use Maclaurin's inequality (see [27], pp 117-119), as described below. Let  $a_1, a_2, \dots, a_n$  be positive real numbers, and for  $k = 1, 2, \dots, n$ , define the averages  $S_k$  as follows:

$$S_k = \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1} a_{i_2} \dots a_{i_k}}{\binom{n}{k}},$$

where the summation is over all distinct sets of  $k$  indices. Maclaurin's inequality then states:

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \dots \geq \sqrt[n]{S_n}.$$

Using Maclaurin's inequality, we thus have:

$$S_1 \geq \sqrt[c/2]{S_{c/2}},$$

and therefore,

$$\sum_{1 \leq i_1 < \dots < i_{c/2} \leq n} a_{i_1} a_{i_2} \dots a_{i_{c/2}} \leq \binom{n}{c/2} \left( \frac{\sum_{j=1}^n a_j}{n} \right)^{c/2}.$$

Now, let  $a_{i_k} = d_k(d_k - 1)$ , where  $d_k$  is the degree of node  $u_k$ . We then have

$$\sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \leq \binom{n}{c/2} \left( \frac{\sum_{u_i \in U} (d_i)(d_i - 1)}{n} \right)^{c/2} \quad (6)$$

$$\begin{aligned} &\simeq \frac{n^{c/2}}{(c/2)!} \left( \frac{\sum_{u_i \in U} (d_i)(d_i - 1)}{n} \right)^{c/2} \\ &= \frac{1}{(c/2)!} \left( 2 \sum_{u_i \in U} \binom{d_i}{2} \right)^{c/2}. \end{aligned} \quad (7)$$

Now, by using (7) in (5) for both sides of the graph, we obtain the following approximation for the expected value of  $N_c$ :

$$\begin{aligned} E(N_c) &\simeq \frac{1}{(c/2)!} \left( 2 \sum_{u_i \in U} \binom{d_i}{2} \right)^{c/2} \frac{1}{(c/2)!} \left( 2 \sum_{w_i \in W} \binom{d'_i}{2} \right)^{c/2} \left( \frac{(\frac{c}{2})! (\frac{c}{2})!}{c} \right) \times \frac{1}{\eta^c} \\ &= \frac{\left( \left( \frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2} \right) \left( \frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2} \right) \right)^{c/2}}{c}. \end{aligned}$$

The proof for showing that  $N_4, \dots, N_k$ , are independent Poisson random variables for any fixed value  $k$  is similar to that of [12] for random non-bipartite graphs and is thus omitted. ■

**Remark 1.** In (6), equality holds if and only if  $d_1 = d_2 = \dots = d_n$ . Thus, in the case of regular degree distributions, approximation (7) is more accurate in comparison with the case of irregular degree distributions, where (7) is an approximate upper bound.

**Corollary 1.** Let  $G = (U \cup W, E)$  be a random bi-regular graph in which all the nodes in  $U$  have the same degree  $d_u$  and all the nodes in  $W$  have the same degree  $d_w$ . Consider the ensemble of such graphs as the number of nodes tends to infinity. In this case, for a fixed even value  $k$ , random variables  $N_4(G), \dots, N_k(G)$ , are independent with Poisson distribution, where the expected value of  $N_c$  is given by

$$E(N_c) \simeq \frac{\left( (d_u - 1)(d_w - 1) \right)^{c/2}}{c}. \quad (8)$$

#### IV. RANDOM LIFTS OF AN ARBITRARY BIPARTITE BASE GRAPH

In this section, we study the cycle distribution of protograph-based LDPC codes that are random lifts of a base graph with no parallel edges. The following result shows that, similar to

random bipartite graphs, for random lifts also, the cycles of different length have independent Poisson distributions.

**Theorem 2.** [25] *For a random  $N$ -lift of a protograph  $G$ , as  $N$  tends to infinity, the distributions of cycles of different length  $c$  tend to independent Poisson distributions with the expected value equal to  $T(G, c)$ , where  $T(G, c)$  is the number of tbc walks of length  $c$  in  $G$ .*

In the following, we calculate  $T(G, c)$  for two special cases of base graphs commonly used in the construction of protograph-based LDPC codes: fully-connected and bi-regular. Although, fully-connected graphs are themselves a special case of bi-regular graphs, in the following, we first consider the case of fully-connected graphs, since for this case, we can in fact, derive an exact expression for  $T(G, c)$ . For the more general case of bi-regular base graphs, our approximation is in the form of an upper bound.

#### A. Calculation of $T(G, c)$ for fully-connected base graphs

**Theorem 3.** *Let  $G = (U \cup W)$  be a fully-connected bipartite graph with  $|U| = a$  and  $|W| = b$ . For any even value  $c \geq 4$ , we have*

$$T(G, c) = \frac{(a-1)(b-1)}{c} \left( (-1)^{c/2} + (a-1)^{c/2-1} \right) \left( (-1)^{c/2} + (b-1)^{c/2-1} \right).$$

*Proof:* To calculate  $T(G, c)$ , we consider the number of tbc walks of length  $c$ ,  $R_{c,e}$ , that go through a specific edge  $e$  in the base graph  $G$ . Due to the symmetry of  $G$ , this number is independent of  $e$ . In the rest of the proof, we thus use the notation  $R_c$  for this number. Since there are  $ab$  edges in  $G$ , we have

$$T(G, c) = \frac{ab \times R_c}{c}, \quad (9)$$

where the division by  $c$  is because each tbc walk is accounted for  $c$  times through its  $c$  edges.

To calculate  $R_c$ , we note that any tbc walk of length  $c$  in the fully-connected based graph can be uniquely described by two interleaving sequences of variable and check nodes, where each sequence corresponds to a closed walk of length  $c/2$  in the complete graph  $K_a$  and  $K_b$ , respectively. Suppose that the number of closed walks of length  $c/2$  starting from a specific node in  $K_a$  is denoted by  $\mathcal{W}_{c/2}^a$ . We thus have

$$R_c = \mathcal{W}_{c/2}^a \times \mathcal{W}_{c/2}^b. \quad (10)$$

To obtain  $\mathcal{W}_k^a$ , we need to calculate a diagonal element of  $A_a^k$ , where  $A_a$  is the  $a \times a$  adjacency matrix of  $K_a$ . It is easy to see that the  $k$ -th power of  $A_a$  has the following general form

$$A_a^k = \begin{pmatrix} \alpha_k & \beta_k & \cdots & \beta_k \\ \beta_k & \alpha_k & \cdots & \beta_k \\ \vdots & \vdots & \ddots & \vdots \\ \beta_k & \beta_k & \cdots & \alpha_k \end{pmatrix},$$

where

$$\alpha_1 = 0, \quad \alpha_{k+1} = (a-1)\beta_k$$

$$\beta_1 = 1, \quad \beta_{k+1} = \alpha_k + (a-2)\beta_k = (a-2)\beta_k + (a-1)\beta_{k-1}.$$

To solve the recursion  $\beta_{k+1} = (a-2)\beta_k + (a-1)\beta_{k-1}$ , we solve the corresponding quadratic equation  $x^2 - (a-2)x - (a-1) = 0$ . The roots of this equation are  $-1$  and  $a-1$ . Thus,  $\beta_k = \gamma(-1)^k + \gamma'(a-1)^k$ . Using  $\beta_1 = 1$  and  $\beta_2 = a-2$ , we obtain  $\gamma' = -\gamma = \frac{1}{a}$ . Hence,  $\beta_k = \frac{-1}{a}(-1)^k + \frac{1}{a}(a-1)^k$ . We thus have

$$\mathcal{W}_k^a = \alpha_k = (a-1)\beta_{k-1} = \frac{a-1}{a}(-1)^k + \frac{1}{a}(a-1)^k. \quad (11)$$

Combining (11) with (10) and (9) completes the proof. ■

**Corollary 2.** *Let  $G = (U \cup W)$  be a fully-connected bipartite graph with  $|U| = a$  and  $|W| = b$ . For any even value  $c \geq 4$ , we have*

$$T(G, c) \approx \frac{\left((a-1)(b-1)\right)^{c/2}}{c}.$$

**Remark 2.** *Combination of Theorem 2 and Corollary 2, and the comparison with the result of Corollary 1 show that, in the asymptotic regime, where the size of the graph tends to infinity, the cycle distributions of random lifts of a fully-connected base graph are identical to those of random bi-regular graphs with the same variable and check node degrees.*

### B. Calculation of $T(G, c)$ for general bi-regular graphs

In this part, we consider the graphs that are bi-regular but not necessarily fully-connected.

**Theorem 4.** *Let  $G = (U \cup W)$  be a bi-regular graph with node degrees equal to  $d_u$  and  $d_w$  on the two sides of the graph, respectively. For any even value  $c \geq 4$ , we have*

$$T(G, c) \leq \frac{|U|d_u}{c} \left( (d_u - 1)(d_w - 1) \right)^{c/2-1}.$$

*Proof:* By counting the tbc walks in  $G$  from the viewpoint of the edges, we have

$$T(G, c) \leq \frac{|U|d_u}{c} K_c, \quad (12)$$

where  $K_c$  is defined as the maximum number of tbc walks of length  $c$  to go through a specific edge in  $G$  (the maximum is taken over all the edges in  $G$ ). Now, for a given edge  $e$  in  $G$ , consider a potential tbc walk in  $G$  that starts from  $e$ . There are  $[(d_u - 1)(d_w - 1)]^{c/2-1}$  possibilities for selecting the following  $c-2$  edges of such a potential tbc walk. For the last edge of the tbc walk, there would be only one choice  $e'$  that can connect the end node of the last edge to the beginning node of  $e$ . This is if such an edge  $e' \neq e$  exists. We thus have  $K_c \leq [(d_u - 1)(d_w - 1)]^{c/2-1}$ . This together with (12) completes the proof.  $\blacksquare$

**Remark 3.** *Note that for a fully-connected base graph, the upper bound of Theorem 4 is approximately equal to the value given in Corollary 2.*

## V. RANDOM CYCLIC LIFTS OF AN ARBITRARY BIPARTITE BASE GRAPH

In this section, we focus on random cyclic liftings of degree  $N$  of a given bipartite base graph  $G$ . The randomness is with respect to the exponent matrix  $P$ , where each non-infinity element of  $P$  is selected in an independent and identically distributed (i.i.d.) fashion from a uniform distribution over  $\mathbb{Z}_N$ . In the following, we first derive upper and lower bounds on the expected value of the number of  $c$ -cycles, followed by an upper bound on the variance.

### A. Calculation of $E(N_c)$

We use the notation  $\mathcal{T}(G, c)$  to denote the set of all tbc walks of length  $c$  in a base graph  $G$ . This set has size  $T(G, c)$ . To derive our results, we need to partition  $\mathcal{T}(G, c)$  into three subsets  $\mathcal{T}_1(G, c)$ ,  $\mathcal{T}_2(G, c)$ , and  $\mathcal{T}_3(G, c)$ . The partition  $\mathcal{T}_1(G, c)$  is the set of all prime ZP tbc walks of length  $c$  in  $G$ , while  $\mathcal{T}_2(G, c)$  consists of all tbc walks  $w$  of length  $c$  in  $G$  such that  $w$  contains at least a ZP tbc subwalk. The partition  $\mathcal{T}_3(G, c)$  covers the rest of the tbc walks of length  $c$  in  $G$ , i.e.,  $\mathcal{T}_3(G, c) = \mathcal{T}(G, c) \setminus (\mathcal{T}_1(G, c) \cup \mathcal{T}_2(G, c))$ .

Consider an edge  $e$  involved in a tbc walk  $w$  in  $\mathcal{T}(G, c)$ . Assume that  $e$  is traversed  $i$  times in one direction and  $j$  times in the opposite direction. The contribution of  $e$  in  $\mathcal{P}(w)$  is thus  $(i - j)p_e$ , where  $p_e$  is the permutation shift of  $e$ . In this case, we say edge  $e$  is of *multiplicity*  $|i - j|$  in  $w$ . We now organize the contribution of different edges of  $w$  in  $\mathcal{P}(w)$  in accordance with their multiplicity, as follows:

$$\mathcal{P}(w) = \sum_{e \in E_1} p_e + 2 \times \sum_{e \in E_2} p_e + \cdots + k \times \sum_{e \in E_k} p_e, \quad (13)$$

where  $E_i$  is the set of edges of multiplicity  $i$ , and  $k$  is the largest multiplicity of edges in  $w$ . In (13), with a slight abuse of notation, we have used  $p_e$  to denote either  $p_e$  or  $-p_e$  depending on the sign of  $i - j$ . In relation to (13), we say tbc walk  $w$  is of *degree*  $k$ . Assuming that  $\ell$  summations (out of  $k$ ) in (13) are non-zero, we refer to  $w$  as a tbc walk of *weight*  $\ell$ . Clearly, ZP tbc walks have both degree zero and weight zero.

**Lemma 1.** *Consider a random cyclic  $N$ -lift of a base bipartite graph  $G$  with no parallel edges, and consider a tbc walk  $w$  of length  $c$  and weight  $\ell \geq 1$  in  $G$ . We then have*

$$\frac{1}{N^\ell} \leq \Pr(\mathcal{P}(w) = 0) \leq \frac{c}{4N}. \quad (14)$$

*Proof:* We first note that the degree  $k$  of a tbc walk of length  $c$  is at most  $c/4$ . This can be easily seen by noting that passing through an edge  $e$ ,  $k$  times, requires passing through  $k$  closed walks, each containing  $e$ . Since graph  $G$  is assumed to have no parallel edges and is bipartite, the length of each such closed walk is at least 4.

For each non-empty set  $E_i$ , the corresponding summation in (13), denoted by  $X_i$ , takes one of the  $N$  values in  $\mathbb{Z}_N$  with equal probability. Also, different summations in (13) are independent, since they share no permutation shifts. The relationship (13) is then a linear integer combination of i.i.d. random variables  $X_i$ 's, and we are interested in evaluating the probability that this linear combination is equal to zero modulo  $N$ . Considering that the weight of  $w$  is  $\ell$ , we are thus interested in the probability that the following equation is satisfied:

$$j_1 X_{j_1} + \cdots + j_\ell X_{j_\ell} = 0 \pmod{N}, \quad (15)$$

where  $j_i, i = 1, \dots, \ell$ , are the indices corresponding to non-zero random variables. The lower

bound of (14) immediately follows by noticing that setting all the random variables equal to zero satisfies (15).

For the upper bound, consider (15), in which all the random variables except  $X_{j_i}$  are fixed. The number of solutions to this equation (considering  $X_{j_i}$  as the variable) is then at most  $\gcd(j_i, N)$ , where  $\gcd(\cdot, \cdot)$  denotes the greatest common divisor. This implies that the probability of (15) being satisfied is upper bounded by  $\gcd(j_i, N)/N$ , and thus by  $\min\{\gcd(j_1, N)/N, \dots, \gcd(j_\ell, N)/N\}$ . Now, the upper bound in (14) follows from  $\gcd(j_i, N) \leq j_i \leq k \leq c/4$ , for any  $j_i$ . ■

**Lemma 2.** *Consider a random cyclic  $N$ -lift of a base bipartite graph  $G$  with no parallel edges, and consider a tbc walk  $w$  in  $\mathcal{T}_1(G, c)$ . We then have*

$$\Pr(A_w) \geq 1 - \frac{c^3}{4N}, \quad (16)$$

where  $A_w$  is the event that none of the subsequences of permutation shifts for  $w$  corresponds to a tbc walk with zero permutation shift.

*Proof:* Denote by  $\bar{A}_w$ , the complement event of  $A_w$ . It is easy to see that the number of subwalks of  $w$  is upper bounded by  $c^2$ . Each such subwalk, based on Lemma 1, is a tbc walk of permutation zero with probability at most  $c/(4N)$ . We thus have  $\Pr(\bar{A}_w) \leq c^3/(4N)$ . This together with  $\Pr(A_w) = 1 - \Pr(\bar{A}_w)$ , completes the proof. ■

**Theorem 5.** *Let  $\tilde{G}$  be a random cyclic  $N$ -lift of a base bipartite graph  $G$  with no parallel edges. For any even value  $c \geq 4$ , we have*

$$(N - \frac{c^3}{4}) \times T_1(G, c) \leq E[N_c(\tilde{G})] \leq N \times T_1(G, c) + \frac{c}{4} \times T_3(G, c),$$

where  $T_i(G, c)$  is the size of the set  $\mathcal{T}_i(G, c)$ .

*Proof:* Consider the base graph  $G$ , and the ensemble of random cyclic  $N$ -lifts  $\tilde{G}$ . The number of cycles of length  $c$  in  $\tilde{G}$  is then given by the following random variable:

$$N_c(\tilde{G}) = N \sum_{w \in \mathcal{T}(G, c)} I(\{\mathcal{P}(w) = 0\} \cap A_w), \quad (17)$$

where  $I(\cdot)$  is the indicator function, and  $A_w$  is the event as defined in Lemma 2. By (17), and using the definition of conditional probability, we have

$$E[N_c(\tilde{G})] = N \sum_{w \in \mathcal{T}(G, c)} \Pr(\{\mathcal{P}(w) = 0\} \cap A_w) = N \sum_{w \in \mathcal{T}(G, c)} \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0). \quad (18)$$

Consider breaking down the summation over the set  $\mathcal{T}(G, c)$  in (18) to summations over the three partitions of  $\mathcal{T}(G, c)$ , i.e., over the sets  $\mathcal{T}_1(G, c)$ ,  $\mathcal{T}_2(G, c)$ , and  $\mathcal{T}_3(G, c)$ . In the following, we evaluate the probability  $\Pr(\{\mathcal{P}(w) = 0\} \cap A_w)$ , for tbc walks  $w$  in the three sets, respectively.

For each tbc walk  $w$  in  $\mathcal{T}_1(G, c)$ , by the definition of  $\mathcal{T}_1(G, c)$ , we have:  $\Pr(\{\mathcal{P}(w) = 0\}) = 1$ . Combining this with Lemma 2, we have

$$1 - \frac{c^3}{4N} \leq \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0) \leq 1. \quad (19)$$

For each tbc walk  $w$  in  $\mathcal{T}_2(G, c)$ , by the definition of  $\mathcal{T}_2(G, c)$ , there is a tbc subwalk  $w'$  of  $w$  such that  $\mathcal{P}(w') = 0$ . So  $\Pr(A_w) = 0$ , and thus

$$\Pr(\{\mathcal{P}(w) = 0\} \cap A_w) = 0. \quad (20)$$

For each tbc walk  $w$  in  $\mathcal{T}_3(G, c)$ , using (14) and  $0 \leq \Pr(A_w | \mathcal{P}(w) = 0) \leq 1$ , we have

$$0 \leq \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0) \leq \frac{c}{4N}. \quad (21)$$

Replacing (19), (20), and (21) in (18) completes the proof. ■

We note that, for a given base graph, the values  $T_1(G, c)$ , and  $T_3(G, c)$ , are fixed with respect to the lifting degree  $N$ . We thus have the following corollary, which demonstrates that the growth of the expected number of  $c$ -cycles with  $N$  can follow two very different trajectories depending on the value of  $c$  and whether the lifted graph has any inevitable cycle of length  $c$  or not.

**Corollary 3.** *Let  $\tilde{G}$  be a random cyclic  $N$ -lift of a base bipartite graph  $G$ . If  $\tilde{G}$  contains inevitable cycles of length  $c$  (i.e., graph  $G$  contains at least one prime ZP tbc walk of length  $c$ ), then, as  $N$  tends to infinity, the expected number of cycles of length  $c$  in  $\tilde{G}$  will be dominated by that of inevitable cycles and grows as  $\Theta(N)$ .<sup>1</sup> On the other hand, if  $\tilde{G}$  contains no inevitable cycles of length  $c$  (i.e., graph  $G$  contains no prime ZP tbc walk of length  $c$ ), then, as  $N$  tends to infinity, the expected number of cycles of length  $c$  in  $\tilde{G}$  is  $\Theta(1)$  (is asymptotically constant with respect to  $N$ ).*

**Remark 4.** *It was shown in [19] that cyclic lifts  $\tilde{G}$  of a base graph with girth  $g$  and no parallel edges have no inevitable cycles of length smaller than  $3g$ . Thus, based on Corollary 3, for  $c < 3g$ , the expected number of cycles of length  $c$  in  $\tilde{G}$  is  $\Theta(1)$ .*

<sup>1</sup>We use the notation  $f(x) = \Theta(g(x))$ , if for sufficiently large values of  $x$ , we have  $a \times g(x) \leq f(x) \leq b \times g(x)$ , for some positive  $a$  and  $b$  values.



**Remark 5.** It is important to note the difference between the expected number of  $c$ -cycles of random lifts, discussed in Section IV, and that of cyclic lifts, discussed in this section. While for random lifts, the expected value is  $\Theta(1)$  with respect to lifting degree  $N$ , regardless of the value of  $c$  or the base graph, for cyclic lifts, it can be  $\Theta(N)$ , depending on the value of  $c$  and the base graph, as explained in Corollary 3.

### B. Calculation of $\text{Var}(N_c)$

In the following, we prove that the variance of the number of cycles of length  $c$  in a random cyclic  $N$ -lift increases at most linearly with  $N$ .

**Theorem 6.** Let  $\tilde{G}$  be a random cyclic  $N$ -lift of a base bipartite graph  $G$  with no parallel edges. As  $N$  tends to infinity, for any fixed even value  $c \geq 4$ , we have

$$\text{Var}[N_c(\tilde{G})] \leq \left( \frac{c^3}{2}T_1^2 + \frac{c}{4}T_3^2 + \frac{c}{2}T_1T_3 \right) \times N + \mathcal{O}(1),^2 \quad (22)$$

where  $T_i$  is used as an abbreviation for  $T_i(G, c)$ .

*Proof:* Following the same notations as in Theorem 5, the number of cycles of length  $c$  in  $\tilde{G}$  is given by the following random variable:

$$N_c(\tilde{G}) = N \sum_{w \in \mathcal{T}(G, c)} I(\{\mathcal{P}(w) = 0\} \cap A_w).$$

We have  $\text{Var}[N_c(\tilde{G})] = E[N_c^2(\tilde{G})] - E^2[N_c(\tilde{G})]$ . In the following, we derive an upper bound on  $E[N_c^2(\tilde{G})]$ . This together with the lower bound on  $E^2[N_c(\tilde{G})]$ , derived in Theorem 5, will prove the theorem. We have

$$\begin{aligned} E[N_c^2(\tilde{G})] &= N^2 \sum_{w \in \mathcal{T}(G, c)} \sum_{w' \in \mathcal{T}(G, c)} E[I(\mathcal{P}(w) = 0 \cap A_w) I(\mathcal{P}(w') = 0 \cap A_{w'})] \\ &= N^2 \sum_{w \in \mathcal{T}(G, c)} \sum_{w' \in \mathcal{T}(G, c)} \Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}). \end{aligned} \quad (23)$$

In the following, for simplicity of notations, we use  $\mathcal{T}$  for  $\mathcal{T}(G, c)$ , and  $\mathcal{T}_i$  for  $\mathcal{T}_i(G, c)$ . To obtain an upper bound on  $E[N_c^2(\tilde{G})]$ , we break each of the two summations in (23) into three, each on one of the three partitions  $\mathcal{T}_1, \mathcal{T}_2$ , and  $\mathcal{T}_3$  of  $\mathcal{T}$ .

<sup>2</sup>The notation  $f(x) = \mathcal{O}(g(x))$  is used, if for sufficiently large values of  $x$ , we have  $|f(x)| \leq a|g(x)|$ , for some positive value  $a$ .

Consider the case where  $w \in \mathcal{T}_1$  and  $w' \in \mathcal{T}_1$ . In this case, we simply use the upper bound of one on  $\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'})$ . This contributes  $T_1^2 \times N^2$  to the upper bound on the variance.

Now consider the case where  $w \in \mathcal{T}_1$  and  $w' \in \mathcal{T}_3$ . In this case, we have

$$\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}) \leq \Pr(\mathcal{P}(w') = 0) \leq \frac{c}{4N}, \quad (24)$$

where the last inequality is from (14). Based on (24), the contribution of this scenario plus the case where  $w \in \mathcal{T}_3$  and  $w' \in \mathcal{T}_1$  in the upper bound is  $c/2 \times T_1 \times T_3 \times N$ . Similarly, based on (24), the contribution of cases where  $w \in \mathcal{T}_3$  and  $w' \in \mathcal{T}_3$  is upper bounded by  $c/4 \times T_3^2 \times N$ .

For all the cases where either  $w$  or  $w'$  is in  $\mathcal{T}_2$ , we have  $\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}) = 0$ , and thus no contribution to the upper bound.

Adding up all the contributions of different cases, as discussed above, we obtain the following upper bound on  $E[N_c^2(\tilde{G})]$ :

$$E[N_c^2(\tilde{G})] \leq T_1^2 \times N^2 + \left(\frac{c}{4}T_3^2 + \frac{c}{2}T_1T_3\right) \times N.$$

This combined with the lower bound of Theorem 5 on  $E^2[N_c(\tilde{G})]$  complete the proof.  $\blacksquare$

## VI. NUMERICAL RESULTS

### A. Random regular and irregular bipartite graphs

In [11], the authors generated random codes from different bi-regular ensembles of LDPC codes, and empirically studied the distribution of cycles of different length in such codes as a function of code's degree distribution and block length. The conclusion of [11] was that the cycle distribution highly depends on the degree distribution but does not change much with the block length. In Section III, we reached a similar conclusion through our theoretical analysis. In the following, we demonstrate through some examples that the expected values that we derived in Theorem 1 and Corollary 1, match the numerical results. We start by the same examples considered in Table IV of [11]. The multiplicities of cycles of different lengths for rate-1/2 bi-regular codes of different degree distributions and lengths are reproduced in Table I here, and compared with the result of Corollary 1. As can be seen, the expected values of Corollary 1 are very close to the cycle multiplicities of random realizations of the graphs for different block lengths, ranging from 200 all the way to 20000.

TABLE I  
MULTIPLICITIES OF SHORT CYCLES IN THE TANNER GRAPHS OF RATE-1/2 RANDOM BI-REGULAR LDPC CODES WITH  
DIFFERENT DEGREE DISTRIBUTIONS AND DIFFERENT BLOCK LENGTHS

Degree Distribution	Short Cycle Distribution	Block Length						$E[N_c]$ Corollary 1
		200	500	1000	5000	10000	20000	
(3,6)	$N_6$	171	167	181	156	166	148	167
	$N_8$	1265	1239	1226	1235	1253	1285	1250
	$N_{10}$	10069	10110	9939	9982	9858	9974	10000
(4, 8)	$N_6$	1636	1611	1584	1562	1537	1572	1544
	$N_8$	25005	24419	24379	24363	24529	24557	24310
	$N_{10}$	409335	409373	408595	407958	408246	409051	408410
(5, 10)	$N_6$	8626	8064	8055	7978	7858	7926	7776
	$N_8$	213639	212484	210767	210153	209614	210159	209952
	$N_{10}$	6052158	6054661	6049148	6043400	6049583	6043704	6046617

As the next example, we consider two irregular degree distributions, and construct random codes of different block length with those degree distributions. The first degree distribution is selected as  $\lambda_I(x) = 0.4286x^2 + 0.5714x^3$ , and  $\rho_I(x) = x^6$ , where the coefficients  $\lambda_i$  and  $\rho_i$  represent the fraction of edges connected to variable and check nodes of degree  $i$ , respectively. This degree distribution, which is mildly irregular, corresponds to an LDPC code with rate 0.5. We thus have  $n = 2m$  for the corresponding Tanner graph, where  $n$  is the block length, and  $m$  is the number of check nodes in the Tanner graph. The second degree distribution is selected from Table I of [28]. It is more irregular than the first degree distribution and is as follows:  $\lambda_{II}(x) = 0.2690x + 0.2603x^2 + 0.0451x^4 + 0.4256x^9$ , and  $\rho_{II}(x) = 0.6398x^6 + 0.3602x^7$ . The code rate corresponding to this degree distribution is 0.4998 [28], and thus  $n \simeq 2m$ . In Table II, we have provided the cycle multiplicities of the random realizations of the two degree distributions at block lengths 200, 500, 1000, 5000, 10000 and 20000, along with the expected values obtained based on Theorem 1. Comparison of the results of Table II with those of Table I show a larger discrepancy between the expected values and the cycle multiplicities in random realizations for irregular graphs vs. regular ones. This can be, at least in part, explained by Remark 1.

TABLE II  
MULTIPLICITIES OF SHORT CYCLES IN THE TANNER GRAPHS OF IRREGULAR LDPC CODES WITH DIFFERENT DEGREE  
DISTRIBUTIONS AND DIFFERENT BLOCK LENGTHS

Degree Distribution	Short Cycle Distribution	Block Length						$E[N_c]$ Theorem 1
		200	500	1000	5000	10000	20000	
$\lambda_I(x), \rho_I(x)$	$N_4$	56	62	61	52	61	59	59
	$N_6$	599	602	587	590	597	602	611
	$N_8$	6653	6814	6742	6881	7011	7158	7067
	$N_{10}$	85244	87260	84846	86436	87046	87311	87181
$\lambda_{II}(x), \rho_{II}(x)$	$N_4$	230	222	244	236	243	196	225
	$N_6$	4871	4759	4057	4571	4562	4769	4500
	$N_8$	109017	107523	104599	106620	105685	107479	101250
	$N_{10}$	2610260	2557357	2212847	2585699	2548117	2605595	2430000

### B. Random lifts of a base graph

As an example, we consider random lifts of the  $3 \times 5$  fully-connected base graph with lifting degrees 400, 1000 and 2000. The cycle multiplicities of the random lifts for cycles of length 4 all the way to 16 are shown in Table III, and compared with the expected value obtained from Theorem 3. As can be seen, for different lifting degrees, the expected value provides a good approximation for the multiplicities of cycles of different length in random realizations.

TABLE III  
MULTIPLICITIES OF SHORT CYCLES OF DIFFERENT LENGTH FOR RANDOM LIFTS OF DIFFERENT DEGREES OF THE  $3 \times 5$   
FULLY-CONNECTED BASE GRAPH

Cycle Length	Lifting Degree			$E[N_c]$ Theorem 3
	$N = 400$	$N = 1000$	$N = 2000$	
4	31	27	29	30
6	64	62	66	60
8	590	588	515	585
10	2994	3111	3083	3060
12	22730	22636	22919	22550
14	147395	148141	147894	147420
16	1058149	1061667	1052401	1056832

### C. Random QC bipartite graphs

In [20], the authors studied the cycle distribution of random cyclic lifts of the  $3 \times 5$  fully-connected base graph for different lifting degrees (block lengths), and observed that the cycle distribution of such graphs is superior to that of random bi-regular codes with the same degree distribution and block length. In particular, a clear improvement in the girth was observed in the experiment. The example also showed that the girth of QC codes was improved by the increase in the lifting degree  $N$ . The above results reported in Table I of [20] are reproduced here in Table IV.

We note that the  $3 \times 5$  fully-connected base graph has girth 4, and thus, based on Remark 4, for  $c \leq 10$ , cyclic random lifts of this base graph have no inevitable cycles of length  $c$ . This means that for  $c \leq 10$ , the expected value of the number of cycles of length  $c$  does not increase with the lifting degree  $N$ . On the other hand, one can find prime ZP tbc walks of length 12, 14 and 16 in the base graph: let  $G = (U, W)$  be the  $3 \times 5$  fully-connected base graph with  $U = \{1, 2, 3\}$  and  $W = \{4, 5, 6, 7, 8\}$ . It is then easy to verify that the following tbc walks in  $G$  are prime with zero permutation shifts:  $w_{12} = 5243514253415$ ,  $w_{14} = 342536143524163$  and  $w_{16} = 25362714263524172$ . This means that the random cyclic lifts of the base graph will have inevitable cycles with these lengths and that, based on Corollary 3, the expected value of cycles with these lengths increases linearly with  $N$  for sufficiently large  $N$  values. These theoretical predictions are consistent with the numerical results reported in Table IV, for these cycle lengths. For cycles of length 18, however, there is no prime ZP tbc walk in the  $3 \times 5$  fully-connected base graph, and thus the expected number of such cycles remains constant with respect to  $N$ . This is also consistent with the results of Table IV.

For comparison, we have also included, in the last column of Table IV, the expected value of the number of cycles in random lifts of the  $3 \times 5$  fully-connected base graph, obtained based on Theorem 3. One can see the large difference between these values and the corresponding values for random cyclic lifts for cases of  $c = 12, 14$ , and 16, where the cyclic lifts have inevitable cycles.

## VII. CONCLUSION

In this paper, we studied the cycle distribution of different ensembles of LDPC codes, often used in the literature, in the asymptotic regime where the block length tends to infinity (but

TABLE IV  
MULTIPLICITIES OF CYCLES OF DIFFERENT LENGTH FOR RANDOM CYCLIC LIFTS OF DIFFERENT DEGREES OF THE  $3 \times 5$   
FULLY-CONNECTED BASE GRAPH

Cycle Length	Lifting Degree			$E[N_c]$
	$N = 400$	$N = 1000$	$N = 2000$	Theorem 3
6	0	0	0	60
8	0	0	0	585
10	2000	1000	0	3060
12	33200	54000	98000	22550
14	193200	275000	478000	147420
16	1022200	1169000	1490000	1056832
18	7143600	7251000	8282000	7427300

the degree distribution is fixed). These ensembles were random irregular and bi-regular, random lifts of protographs, and random cyclic lifts of protographs. We demonstrated that for the first ensemble, the multiplicities of cycles of different lengths have independent Poisson distributions with the expected values only a function of cycle length and degree distributions, and independent of the block length. We also showed that for the second ensemble, the asymptotic cycle distributions have the same behavior as those of the first ensemble as long as the degree distributions are identical. For the third ensemble, we proved that the cycle distributions can be significantly different than those of the first two ensembles. In particular, we showed that for some values of  $c$ , and depending on the protograph, the expected number of  $c$ -cycles can increase linearly with the block length. We also derived an upper bound, linearly increasing with the block length, on the variance of the number of  $c$ -cycles. Using numerical results, we also demonstrated that our asymptotic results provide good approximations for the number of cycles in realizations of finite-length LDPC codes, even when the block length is as short as a few hundred bits. Moreover, our results provided theoretical justification for some of the observations made empirically in the literature about cycle distributions of LDPC codes.

The results presented in this paper can be used in estimating the multiplicities of other important substructures of LDPC Tanner graphs that consist of cycles, such as trapping sets. They can also be used in the analysis and design of LDPC codes in cases where such processes depend on the knowledge of the cycle distributions.

## REFERENCES

- [1] Y. Mao and A. H. Banihashemi, "A heuristic search for good low-density parity-check codes at short block lengths," in *Proc. IEEE Int. Conf. Commun.*, vol. 1, Helsinki, Finland, Jun. 2001, pp. 41–44.
- [2] X.-Y. Hu, E. Eleftheriou, and D. M. Arnold, "Regular and irregular progressive edge-growth Tanner graphs," *IEEE Trans. Inform. Theory*, vol. 51, no. 1, pp. 386–398, Jan. 2005.
- [3] T. R. Halford and K. M. Chugg, "An algorithm for counting short cycles in bipartite graphs," *IEEE Trans. Inform. Theory*, vol. 52, no. 1, pp. 287–292, Jan. 2006.
- [4] H. Xiao and A. H. Banihashemi, "Error rate estimation of low-density parity-check codes on binary symmetric channels using cycle enumeration," *IEEE Trans. Communications*, vol. 57, no. 6, pp. 1550–1555, Jun. 2009.
- [5] R. Asvadi, A. H. Banihashemi, and M. Ahmadian-Attari, "Lowering the error floor of LDPC codes using cyclic liftings," *IEEE Trans. Inform. Theory*, vol. 57, no. 4, pp. 2213–2224, Apr. 2011.
- [6] M. Karimi and A. H. Banihashemi, "On characterization of elementary trapping sets of variable-regular LDPC codes," *IEEE Trans. Inform. Theory*, vol. 60, no. 9, pp. 5188–5203, Sep. 2014.
- [7] M. Karimi and A. H. Banihashemi, "Efficient algorithm for finding dominant trapping sets of LDPC codes," *IEEE Trans. Inform. Theory*, vol. 58, no. 11, pp. 6942–6958, Nov. 2012.
- [8] Y. Hashemi and A. H. Banihashemi, "On characterization and efficient exhaustive search of elementary trapping sets of variable-regular LDPC codes," *IEEE Commun. Lett.*, vol. 19, no. 3, pp. 323–326, March 2015.
- [9] Y. Hashemi and A. H. Banihashemi, "New characterization and efficient exhaustive search algorithm for leafless elementary trapping sets of variable-regular LDPC codes," *IEEE Trans. Inform. Theory*, available online: <https://arxiv.org/abs/1510.04954>.
- [10] J. Flum and M. Grohe, "The parameterized complexity of counting problems," *SIAM J. Comput.*, vol. 33, no. 4, pp. 892–922, 2004.
- [11] M. Karimi and A. H. Banihashemi, "Message-passing algorithms for counting short cycles in a graph," *IEEE Trans. Communications*, vol. 61, no. 2, pp. 485–495, Feb. 2013.
- [12] B. Bollobás, "A probabilistic proof of an asymptotic formula for the number of labelled regular graphs," *European J. Combin.*, vol. 1, no. 4, pp. 311–316, Dec. 1980.
- [13] B. D. McKay, N. C. Wormald, and B. Wysocka, "Short cycles in random regular graphs," *Electron. J. Combin.*, vol. 11, no. 1, p. 66, 2004.
- [14] J. Thorpe, "Low-density parity-check (LDPC) codes constructed from protographs," *IPN progress report 42–154*, JPL, Aug. 2003.
- [15] S. Abu-Surra, D. Divsalar, and W. E. Ryan, "Enumerators for protograph-based ensembles of LDPC and generalized LDPC codes," *IEEE Trans. Inform. Theory*, vol. 57, no. 2, pp. 858–886, Jan. 2011.
- [16] L. Dolecek, D. Divsalar, Y. Sun, and B. Amiri, "Non-binary protograph-based LDPC codes: enumerators, analysis, and designs," *IEEE Trans. Inform. Theory*, vol. 60, no. 7, pp. 3913–3941, Apr. 2014.
- [17] M. P. C. Fossorier, "Quasi-cyclic low-density parity-check codes from circulant permutation matrices," *IEEE Trans. Inform. Theory*, vol. 50, no. 8, pp. 1788–1793, Aug. 2004.
- [18] S. Myung, K. Yang, and J. Kim, "Quasi-cyclic LDPC codes for fast encoding," *IEEE Trans. Inform. Theory*, vol. 51, no. 8, pp. 2894–2901, Aug. 2005.
- [19] S. Kim, J.-S. No, H. Chung, and D.-J. Shin, "Quasi-cyclic low-density parity-check codes with girth larger than 12," *IEEE Trans. Inform. Theory*, vol. 53, no. 8, pp. 2885–2891, Aug. 2007.

- [20] M. Karimi and A. H. Banihashemi, "Counting short cycles of quasi cyclic protograph LDPC codes," *IEEE Commun. Lett.*, vol. 16, no. 3, pp. 400–403, Mar. 2012.
- [21] K.-J. Kim, J.-H. Chung, and K. Yang, "Bounds on the size of parity-check matrices for quasi-cyclic low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 59, no. 11, pp. 7288–7298, Aug. 2013.
- [22] M. Karimi and A. H. Banihashemi, "On the girth of quasi-cyclic protograph LDPC codes," *IEEE Trans. Inform. Theory*, vol. 59, no. 7, pp. 4542–4552, Jul. 2013.
- [23] IEEE-802.11n, Wireless LAN Medium Access Control and Physical Layer Specifications: Enhancements for Higher Throughput, P802.11n/D3.07, Mar. 2008
- [24] Amendment: Physical Layer and Management Parameters for 10Gb/s Operation, Type 10GBASE-T, IEEE Draft P802.3an/D2.1.
- [25] J.-P. Fortin and S. Rudinsky, "Asymptotic eigenvalue distribution of random lifts," *The Waterloo Mathematics Review*, vol. 2, no. 2, pp. 1–10, Oct. 2012.
- [26] J. L. Gross and T. W. Tucker, *Topological graph theory*. NewYork, NY, USA: Wiley, 1987.
- [27] Z. Cvetkovski, *Inequalities*. Springer, Heidelberg, 2012.
- [28] H. Saeedi and A. H. Banihashemi, "On the design of LDPC code ensembles for BIAWGN channels," *IEEE Trans. Communications*, vol. 58, no. 5, pp. 1376–1382, May 2010.